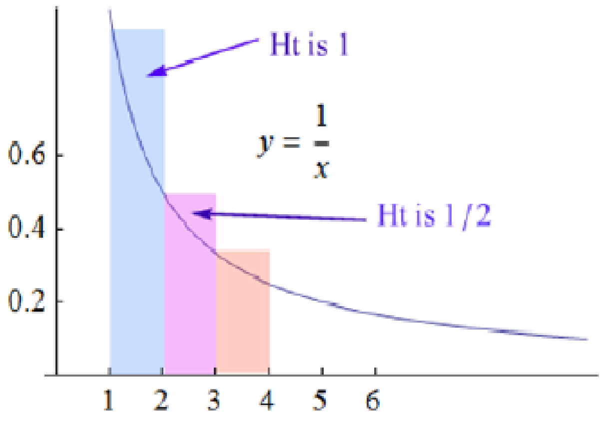
المحاضرة الرابعة

Convergence and Divergence of Series

The Key Question- does the series converge?

Example: Show that the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.



The sum of the areas of all these rectangles is

$$1+\frac{1}{2}+\frac{1}{3}+\cdots$$

The area under the curve $\int_1^\infty \frac{1}{x} dx$ is smaller than the sum of the areas of the rectangles.

$$\int_1^\infty \frac{1}{x} dx = \ln x \Big]_1^\infty = \ln\infty - \ln 1 = \infty .$$

The area under the curve *y = 1/x* is less than the sum of areas of rectangles

On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the integral $\int_1^\infty \frac{1}{x^2} dx$ is finite.

The integral Test

Let $\{a_n\}$ be a sequence of positive numbers. Let $a_n = f(n)$ where $f(x)$ is a continuous, positive, and decreasing function for all $x \geq N$ where N is a positive integer.

Then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_N^\infty f(x) dx$ both converge or both diverge.

Since $\int_1^\infty \frac{1}{x} dx = \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ must diverge.
This illustrates that we can use an integral to test if a series converges.

Example:

Use Integral Test to determine whether or not $\sum_{n=1}^{\infty} \frac{4}{n^3}$ converges.

Note that if n is raised to a high enough power, the series will converge.

To see if $\sum_{n=1}^{\infty} \frac{4}{n^3}$ converges, determine whether or not $\int_1^{\infty} \frac{4}{x^3} dx$ converges.

$$\int_1^{\infty} 4x^{-3} dx = \frac{4x^{-2}}{-2} = \frac{2}{x^2}\Big]_{\infty}^{1} =$$

$$\frac{2}{1} - \frac{1}{\infty} = 2 - 0 = 2\,.$$

Since the integral converges, the sum converges.